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Heisenberg's relations in discrete N-dimensional parameterized metric vector spaces

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This paper shows that Heisenberg's relations are deducible from the structure of parameterized metric vector spaces of arbitrary dimension, by means of some new ideas, not entirely found in the current and vast literature about this subject. In order to allow this task to be done, some new concepts are put forward, as the *real inward product of two vectors*, described within vector spaces defined over the complex field, which permit further construction of real scalar products and Euclidian norms. Also, the definition of *triads* in *parameterized metric vector spaces*, which are constructs formed by three linearly independent vectors: a parameterized vector, the *triad generator*, and a pair of vectors orthogonal to the generator, the *triad companions*, one is simply made of the generator multiplied by the parameter and the other by using the first derivative with respect the parameter over the generator. Triads appear as forming a crucial building block for deduction of Heisenberg's relations. Building Heisenberg's relations is based on setting up the Gram matrix of a vector pair, which provides a new definition of the cosine of the angle subtended by two vectors in complex metric spaces, as well as an ancillary redefinition of the Schwarz inequality. It must be noted the fact that covariance of conjugated pairs of quantum mechanical variables appears to be a constant in the present scheme and others, a property which seems to become equivalent to the Heisenberg's relations, where in the current descriptions only variances or uncertainties are involved.

KEY WORDS: Heisenberg's relations, N-dimensional vector spaces, parameterized metric vector spaces, inward vector product, real scalar product, Gram matrix, Schwartz inequality

1. Introduction

In one hand, Bunge [1] cleverly revises the role of mathematics in science, technology and humanities. It is within this philosophical spirit that the present study has been sketched, after a long elaboration, as merits the crucial role Heisenberg's relations [2] play on quantum mechanics. In the other hand, the idea triggering the present development shall be found in recently

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published theoretical advances, dealing with the interpretation of the connection between classical and quantum mechanics. This mentioned study points out towards Heisenberg's relations as constituting a cornerstone of quantum theory [3], from where Schrödinger's equation can be deduced.

It will be foolish to compete with the treatment of Heisenberg's relations made by such brilliant discussions, as those proposed in the books of Bohm [4], Dirac [5], Margenau and Murphy [6], Messiah [7], Schwinger [8], von Neumann [9] and Weyl [10] among many others, which will not be cited for brevity. Instead, the aim of this study is to show how by constructing simple vector space operations, proposing a few set of definitions and setting a succinct collection of axioms, the Heisenberg's uncertainty relations can be deduced from the structure of any appropriately chosen *complex parameterized metric vector space*.

In order to achieve this proposal, the present work will be organized in the following manner: First a discussion on how real vector spaces can be constructed from complex ones will be put forward. Next the definition of a real valued scalar product within complex vector spaces will be described. This will permit to redefine at once the *Gram matrix* of a vector set; hence the *Gramian* of the same vector set. From there it can be deduced the concept of *cosine of the angle subtended by two vectors*, and next the ancillary *Schwartz inequality* can be obtained as well. Such preliminary study will permit to describe *parameterized metric vector spaces*, in the same way as they were used in a recent description of space–time frames of arbitrary dimension [11]. The structure of *triads* in parameterized metric vector spaces is proposed next. In a triad three linearly independent vectors are chosen: first an arbitrary *generator vector*: $|x\rangle$, is set, completing this structure with two attached *companion vectors*. The companion vectors are also supposed to belong to the same parameterized metric space, and are made by the generator multiplied by the basic parameter: $|tx\rangle$ and the first derivative of the generator with respect to this same parameter: $|\partial x/\partial t\rangle$. Both companion vectors can be easily seen that they can be made *orthogonal* to the generator, but not necessarily among themselves, a fact which appears later on to be closely related to Heisenberg's relations. After these preliminary settings, the description of the *expectation values* in parameterized metric vector spaces leads finally to the deduction of Heisenberg's relations as a natural property emanating from such space structure.

2. Construction of real spaces from complex vector spaces

Suppose known an N-dimensional vector space over the *complex* field: $V_N(\mathbf{C})$. A new space defined over the *real* field, which can be considered a subspace of the former structure, can be easily imagined, employing a *real inward vector product*: $\{x : y\}$, defined as follows:

$$\forall x, y \in V(\mathbf{C}) \rightarrow \exists z = \{x : y\} = \frac{1}{2}\left(x^* * y + y^* * x\right) \in V(\mathbf{R}) \subset V(\mathbf{C}).$$

As the above defined average operation is nothing else, but the real part of the involved inward products, it can be also written as

$$\{x : y\} = \mathrm{Re}(x * y) \in V(\mathbf{R}).$$

The operation: $\{x : y\}$, when applied among the elements of a vector space defined over the real field: $V(\mathbf{R})$, simply turns into the *inward vector product* [12, 13] of two vectors:

$$\forall x, y \in V(\mathbf{R}) : z = \{x : y\} = x * y \rightarrow \forall I : z_I = x_I y_I \wedge z \in V(\mathbf{R}).$$

3. Real scalar product of two vectors

The above definition, the inward real product of two vectors: $\{x : y\}$, permits to redefine a scalar product in $V(\mathbf{C})$ producing a *real* number:

$$\forall x, y \in V(\mathbf{C}) : \langle x : y\rangle = \langle\{x : y\}\rangle \in \mathbf{R},$$

where has been used the whole vector elements *summation symbol* [14]:

$$z = \{z_I\} \in V(\mathbf{R}) : \langle z\rangle = \sum_I z_I \in \mathbf{R}.$$

Within these previous definitions it can be seen that the set of self-transformed vectors: $\{x : x\}$ produces the elements of a vector *semispace*: $V\left(\mathbf{R}^+\right)$ [15]. That is: the set of vectors with real positive elements. Also the vector Euclidean norms can be related to this construction, as one can write

$$\langle x|x\rangle = \langle x : x\rangle = \langle\{x : x\}\rangle = \sum_I |x_I|^2 \in \mathbf{R}^+.$$

4. Gram matrix and Gramian of two vectors

Although the *Gram matrix* [16] of two vectors is defined by means of the classical scalar products, there one can envisage a parallel definition using the real scalar products construction as defined above. The symmetric Gram matrix will be written in this case as

$$G(x : y) = \begin{pmatrix} \langle x : x\rangle & \langle x : y\rangle \\ \langle y : x\rangle & \langle y : y\rangle \end{pmatrix} = \begin{pmatrix} \langle x|x\rangle & \langle x : y\rangle \\ \langle x : y\rangle & \langle y|y\rangle \end{pmatrix}.$$

The *Gramian* [16], the determinant of the Gram matrix, is readily seen to have the form

$$\Gamma(x : y) = \mathrm{Det}|G(x : y)| = \langle x|x\rangle\langle y|y\rangle - |\langle x : y\rangle|^2,$$

and the linear independence of the vector pair can be associated to a non-negative form of the Gramian:

$$\langle x|x\rangle\langle y|y\rangle - |\langle x : y\rangle|^2 \geqslant 0,$$

which, in turn, produces the equivalent *Schwartz inequality* [16]:

$$\langle x|x\rangle\langle y|y\rangle \geqslant |\langle x : y\rangle|^2. \tag{1}$$

To obtain the usual forms of the Gram matrix, the Gramian and the Schwartz inequality, there is only necessary to perform the substitution: $\langle x : y\rangle \rightarrow \langle x|y\rangle$.

5. Cosine of the angle subtended by two vectors and the associated Schwartz inequality

Then, owing to these previous considerations, the *cosine of the angle subtended by two vectors*, $\cos(\alpha)$, [16] can be simply redefined, using the previous real scalar product definition, by means of the algorithm:

$$\cos(\alpha) = \langle\{x : y\}\rangle(\langle\{x : x\}\rangle\langle\{y : y\}\rangle)^{-1/2},$$

which, employing the previous equivalent definitions, can be also written as[1]

$$\cos(\alpha) = \langle x : y\rangle\,(\langle x : x\rangle\langle y : y\rangle)^{-1/2} = \langle x : y\rangle\,(\langle x|x\rangle\langle y|y\rangle)^{-1/2}.$$

Thus, producing a more familiar picture connected with this measure of the angle between two vectors [5].

By definition of the involved scalar products, in general, for the $\cos(\alpha)$ function the following range will be obtained:

$$-1 \leqslant \cos(\alpha) \leqslant +1,$$

which indicates that an equivalent expression to Schwartz inequality can be easily written in absolute value

$$|\langle x : y\rangle| \leqslant (\langle x|x\rangle\langle y|y\rangle)^{-1/2}, \tag{2}$$

[1]Note that the usual expression, involving whole scalar products, is written employing scalar product module, as

$$\cos_0(\alpha) = |\langle x|y\rangle|\,(\langle x|x\rangle\langle y|y\rangle)^{-1/2}.$$

which tells that the alternative squared form (1) can be also used. Therefore, the cosine of the angle subtended by two vectors and the Schwartz inequality are defined in the same context. Therefore the definition of the cosine of the angle subtended by two vectors is related to the non-negative definiteness of the Gramian of such vector pair.

6. Variance and covariance

As it will be a basic matter for further mathematical development, the definitions of *variance* and *standard deviation* [17] of a vector, under some restrictions, will be now discussed. Suppose that the components of the involved vectors, in the frame of the previously defined real scalar products, have a *null arithmetic mean*. That is, using the vector components summation symbol:

$$\langle x \rangle = \langle y \rangle = 0.$$

If this is the case, the involved scalar products can be without difficulty associated to the variance of the vector elements, so it can be written for the vector x:

$$\mathrm{var}(x) = \sum_I |x_I|^2 = \langle x|x \rangle = \langle x : x \rangle$$

with a similar definition for $\mathrm{var}(y)$. Following the same trend, the alternative definition for the covariance of two vectors can be written as the real scalar product:

$$\mathrm{cov}(x; y) = \sum_I \mathrm{Re}\left(x_I^* y_I\right) = \langle x : y \rangle.$$

Owing to these properties, then one can write:

$$\forall x, y \in V(\mathbf{C}) : \langle x \rangle = 0 \wedge \langle y \rangle = 0 \rightarrow \cos(\alpha) = \mathrm{cov}(x; y)\left(\mathrm{var}(x)\mathrm{var}(y)\right)^{-1/2},$$

which is a well-known statistical result [17]. And owing to the Schwarz inequality form, there can also be written under the zero mean restriction:

$$|\mathrm{cov}(x; y)| \leqslant (\mathrm{var}(x)\mathrm{var}(y))^{-1/2}. \quad (3)$$

7. Parameterized metric vector spaces

Suppose, as it has been done in a previous study on space–time frames [11], that the elements of the initial complex vector space studied: $V(\mathbf{C})$, are, in addition, well-defined functions of some real parameter: t.[2] Symbolizing these spaces by: $V(\mathbf{C}; t)$. That is, in general it can be written the following definition:

$$\forall x \in V(\mathbf{C}; t) \to x = \{x_I(t) \ |t \in D \subseteq \mathbf{R} \wedge x_I(t) \in \mathbf{C}\}.$$

Then, the previous real scalar product definition applies as well in these *parameterized metric vector spaces*. However, a slight integration add-in envelope shall be added, which has to be related to the presence of the parameter, when norms and scalar products are involved, that is, for instance,

$$\begin{aligned}\forall x \in V(\mathbf{C}; t) \to \langle x : x\rangle &= \langle x|x\rangle \\ &= \sum_I \left(\int_D |x_I(t)|^2 \,\mathrm{d}t\right) = \int_D \left(\sum_I |x_I(t)|^2\right) \mathrm{d}t \\ &= \sum_I \langle x_I(t)|x_I(t)\rangle = \left\langle \sum_I |x_I(t)|^* \sum_I |x_I(t)| \right\rangle,\end{aligned}$$

where use of the *Hadamard product* [13] between two sums has been employed in the last term [3] of the equality sequence, along the usual scalar product formalism employed in functional spaces.

8. Triads in parameterized metric vector spaces

When in a parameterized metric vector space an arbitrary vector is chosen, it can be also admitted the axiom consisting into that two attached vectors, defined as follows, also belong to the same space. Taking this into account, it can be admitted that:

$$\forall x(t) \in V(\mathbf{C}; t) \to \exists \left\{ t x(t) \wedge \frac{\partial}{\partial t}[x(t)] \right\} \in V(\mathbf{C}; t), \tag{4}$$

holds, where the parameter product and the first derivative vectors are defined as

$$t x(t) = \{t x_I(t)\} \wedge \frac{\partial}{\partial t}[x(t)] = \left\{ \frac{\partial x_I(t)}{\partial t} \right\}.$$

[2]Here and in the following paper development, a monodimensional parameter feature will be used for simplicity, but the reader can grasp immediately that the results can be extended to encompass any parametric dimension. This is the reason why, along the development of this paper, the partial derivative symbol $\partial/\partial t$ has been used to note derivation with respect the space parameter t.

[3]The Hadamard product involving two sums, possessing an equal number of terms, is defined as

$$\left(\sum_I a_I\right)\left(\sum_I b_I\right) = \sum_I a_I b_I.$$

Furthermore, any triad defined over $V(\mathbf{C};t)$, defined as in equation (4): $T(x) = \{x; tx; \partial x/\partial t\}$ can be considered, in general, as a set of three linearly independent vectors, and, as such, the two former ones, which we will call thereafter *triad companions*, can be considered orthogonal to the first, which will be called the *triad generator*. That is, the following relations are supposed to hold within a triad:

$$x \perp tx \wedge x \perp \frac{\partial x}{\partial t}.$$

Such an orthogonal condition between triad members can be simply constructed. First, by supposing that the triad generator vector is normalized, and then admitting that a *Gram–Schmidt orthogonalization process* [16] has also taken place with the first companion vector in the usual form as

$$tx \rightarrow |tx\rangle - \langle x|tx\rangle|x\rangle = |tx\rangle - |x\rangle\langle x|tx\rangle = (I - P_x)|tx\rangle = P_{\tilde{x}}|tx\rangle.$$

Finally, one can employ a similar process for the derivative vector, that is

$$\frac{\partial x}{\partial t} \rightarrow P_{\tilde{x}}\left|\frac{\partial x}{\partial t}\right\rangle,$$

where $P_x = |x\rangle\langle x|$, is just the projection operator over the generator vector and, thus, the operator: $P_{\tilde{x}} = I - P_x$ corresponds to the orthogonal complement projector: $P_{\tilde{x}}|x\rangle = |0\rangle$. As a result, the initial companion vector: $|tx\rangle$ can be substituted by the vector: $P_{\tilde{x}}|tx\rangle$ and the initial first parameter derivative companion: $|\partial x/\partial t\rangle$ by $P_{\tilde{x}}|\partial x/\partial t\rangle$ as well.

Such a situation can be visualized schematically in figure 1.

However, in order to ease the notation it will be considered thereafter that: $|tx\rangle \equiv P_{\tilde{x}}|tx\rangle$, and the same will be supposed to be true for the first derivative companion vector. This is the same as to admit that, from now on, the following orthogonal relations will always hold:

$$\langle x|tx\rangle = 0 \wedge \left\langle x\left|\frac{\partial x}{\partial t}\right.\right\rangle = 0. \tag{5}$$

The triad construction in parameterized vector spaces is related with the recent development of the extended wave functions formalism [18], where the wave function and its gradient are taken as elements of a vector, forming in this way the primary structure of the extended wave function. Extended wave functions can be defined with further refinement and extended, in such a way that non-linear Schrödinger equations naturally appear.

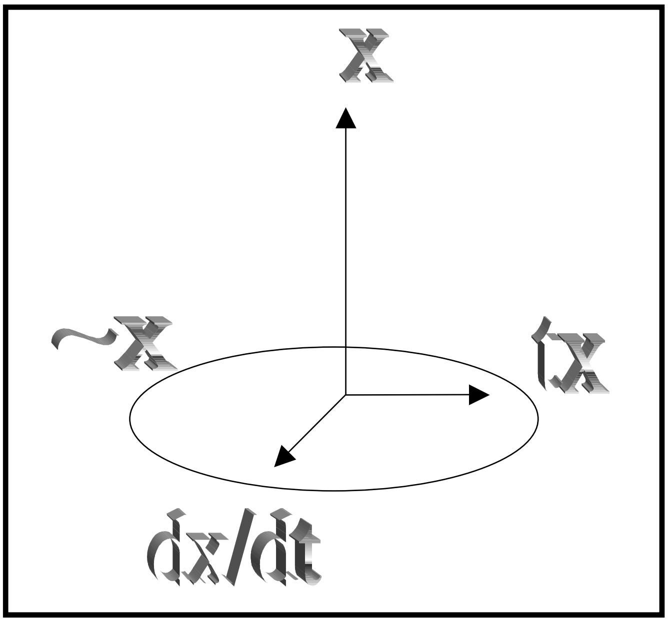

Figure 1. Orthogonal relationship between the vectors composing the triad $T(x)$.

9. Expectation values in parameterized metric spaces

The above defined orthogonal relations of equation (5), being fulfilled within a parameterized vector triad, can further lead to the definitions of the following *expectation values*, which can be formally defined as

$$\langle t\rangle = \langle x|t|x\rangle = \langle x|tx\rangle = \langle tx|x\rangle = 0 \tag{6}$$

and also

$$\left\langle\frac{\partial}{\partial t}\right\rangle = \left\langle x\left|\frac{\partial}{\partial t}\right|x\right\rangle = \left\langle\frac{\partial x}{\partial t}\right|x\right\rangle = \left\langle x\left|\frac{\partial x}{\partial t}\right.\right\rangle = 0, \tag{7}$$

which are entirely a consequence of the orthogonal relations (5), supposedly associated to $T(x)$ triad companions of the generator vector. This implicit simplification, due to the orthogonal relations (5) has been already employed by Weyl [10].

Therefore, the following attached variances may be now written as

$$\operatorname{var}(t) = \langle t^2\rangle = \langle x|t^2|x\rangle = \langle tx|tx\rangle$$

and also for the derivative companion:

$$\operatorname{var}\left(\frac{\partial}{\partial t}\right) = \left\langle\frac{\partial^2}{\partial t^2}\right\rangle,$$

where in the above expression it can be supposed, or better taken axiomatically, as an additional intrinsic parameterized space property [4] of the vector components, that one of the possible equalities hold:

$$\left\langle \frac{\partial x}{\partial t} | \frac{\partial x}{\partial t} \right\rangle = \pm \left\langle x \left| \frac{\partial^2}{\partial t^2} \right| x \right\rangle,$$

and in case the negative sign is active, there appears the usual Green theorem well-known property [6]. This poses no great problem from the quantum mechanical side, as the implied functions have to possess precise characteristic properties fulfilling this characteristic feature, related to the kinetic energy representation [19].

The variances, associated to the triad companion vectors, are in this construction equal to their Euclidian norms, and so, they can be used as components of the Schwartz inequality structure, as already described in equation (3).

10. Heisenberg's relations in parameterized N-dimensional metric vector spaces

Using instead the modified Schwartz inequality as set in equation (1), one can write

$$\left\langle tx : \frac{\partial x}{\partial t} \right\rangle^2 \leqslant \langle tx|tx \rangle \left\langle \frac{\partial x}{\partial t} \middle| \frac{\partial x}{\partial t} \right\rangle,$$

which, owing to the previous variance description, can be also alternatively written as

$$\left\langle tx : \frac{\partial x}{\partial t} \right\rangle^2 \leqslant \left\langle t^2 \right\rangle \left\langle \frac{\partial^2}{\partial t^2} \right\rangle. \tag{8}$$

The left-hand side term can be related to the unit norm of the initial vector, that is: $\langle x|x \rangle = 1$, as admitted in the triad definition, and it can be also written as

$$\begin{aligned}
\left\langle \frac{\partial}{\partial t}\{x : tx\} \right\rangle &= \left\langle \left\{ \frac{\partial}{\partial t} x : tx \right\} \right\rangle + \langle \{x : x\} \rangle + \left\langle \left\{ x : t \frac{\partial}{\partial t} x \right\} \right\rangle \\
&= \left\langle \frac{\partial x}{\partial t} : tx \right\rangle + \langle x|x \rangle + \left\langle tx : \frac{\partial x}{\partial t} \right\rangle \\
&= 2 \left\langle \left\{ tx : \frac{\partial x}{\partial t} \right\} \right\rangle + \langle x|x \rangle \\
&= 2 \left\langle tx : \frac{\partial x}{\partial t} \right\rangle + \langle x|x \rangle.
\end{aligned} \tag{9}$$

[4]This is the same as to associate to the parameter functions, composing the vector elements of the space: $V(\mathbf{C}; t)$, such a property.

Taking into account that it may be assumed, as in the Messiah discussion [7] or proved by Weyl arguments [10], or simply accepted as an additional vector element axiomatic property, that the left-hand side term is null:

$$\left\langle \frac{\partial}{\partial t}\{x : tx\} \right\rangle = \int_D \frac{\partial}{\partial t}\langle\{x : tx\}\rangle \, \mathrm{d}t = 0, \tag{10}$$

then, from equation (9) and the normalization of the triad generator vector it is found:

$$-\left\langle tx : \frac{\partial x}{\partial t} \right\rangle = \frac{1}{2}\langle x|x\rangle = \frac{1}{2}, \tag{11}$$

therefore it is readily obtained:

$$\left\langle tx : \frac{\partial x}{\partial t} \right\rangle^2 = \frac{1}{4}, \tag{12}$$

so one can write equation (8) in the form

$$\frac{1}{4} \leqslant \langle t^2 \rangle \left\langle \frac{\partial^2}{\partial t^2} \right\rangle, \tag{13}$$

which constitutes a *general Heisenberg's relationship*, as quoted by Schwinger [8], holding in N-dimensional parameterized metric vector spaces.

It must be also noted that, in this case, one can also associate the left-hand side real scalar product of equation (12) with the covariance between the triad companion vectors as

$$\operatorname{cov}(t; \frac{\partial}{\partial t}) = \left\langle tx : \frac{\partial x}{\partial t} \right\rangle, \tag{14}$$

due to the expectation value properties obtained in equations (6) and (7), which in turn constitutes a similar result as the one discussed by Bohm [4].

Of course, equation (13) can be also alternatively written as

$$\frac{1}{4} \leqslant \operatorname{var}(t) \operatorname{var}\left(\frac{\partial}{\partial t}\right) \tag{15}$$

and equation (15) can be rearranged in an even more usual formulation, employing the standard deviations, or uncertainties, associated to both triad companion vectors:

$$\Delta(t) = [\operatorname{var}(t)]^{1/2} \wedge \Delta\left(\frac{\partial}{\partial t}\right) = \left[\operatorname{var}\left(\frac{\partial}{\partial t}\right)\right]^{1/2}$$

equation (13) taking finally, in this manner, the alternative widespread literature form:

$$\frac{1}{2} \leqslant \Delta(t)\Delta\left(\frac{\partial}{\partial t}\right). \tag{16}$$

11. Constancy of the triad companions covariance

Relations (11) and (12) seem to tell more than a simple equality is fulfilled. The interesting thing to note here consists into the fact that, from these equations it can be deduced the two triad companions cannot be, in any circumstance, orthogonal, but have a precise non-null value of the cosine of the subtended angle. This also means that if the cosine of their subtended angle could never reach the zero value, then the Heisenberg's relations essential feature has to be contained in this property of the triad companion vectors. From the statistical point of view it seems that it can be deduced from equations (12) and (14) that:

$$\mathrm{cov}(t; \frac{\partial}{\partial t}) = \pm\frac{1}{2}$$

or what is the same: the covariance of the triad companion vectors has to be always a constant, whenever equation (10) is assumed to hold for the parametric functions. In quantum mechanical terms, thinking of position and linear momentum, or time and energy, such a result means that the covariance of a conjugated pair of quantum mechanical variables, acting as triad companions, is, in any case, a non-null constant. Perhaps this could be seen as an *alternative* formulation of Heisenberg's relations. It has been Bohm [4] again, several years ago, who noticed the important role of covariance and its generalization in quantum mechanics.

12. Alternative Heisenberg's relations

Alternatively, one can consider the possibility to accept that a more flexible situation than the one expressed in equation (10) and accepted as an axiom can be envisaged. For this purpose it is just sufficient to choose another axiom stating the integral derivative (9) has to be convergent, which is the same as to admit it can take without problems the form

$$\left\langle \frac{\partial}{\partial t}\{x : tx\} \right\rangle = \omega,$$

where the constant ω has to be finite but $\omega \neq 1$, then one will have

$$\left\langle tx : \frac{\partial x}{\partial t} \right\rangle = \frac{1}{2}(\omega - \langle x|x\rangle) = \frac{1}{2}(\omega - 1) = \frac{1}{2}\lambda,$$

and with the constant $\lambda \neq 0$ but finite. So, finally, the associated Heizenberg's relations could be written in the variance form

$$\frac{1}{4}\lambda^2 \leqslant \mathrm{var}(t)\mathrm{var}\left(\frac{\partial}{\partial t}\right) \tag{17}$$

or the equivalent uncertainty relationship may be written as

$$\frac{1}{2}\lambda \leqslant \Delta(t)\Delta\left(\frac{\partial}{\partial t}\right). \tag{18}$$

Also, taking again into consideration the role of the covariance, then in this situation one will have have insead

$$\mathrm{cov}(t; \frac{\partial}{\partial t}) = \pm\frac{1}{2}\lambda, \tag{19}$$

then the same considerations as these provided in the particular case $\lambda = 1$, can apply here: equation (19) is equivalent to Heisenberg's relations written in the form (17) or (18).

13. Conclusions

It has been demonstrated that, under some elementary and reasonable space settings, Heisenberg's relations appear to be a deducible general property of the parameterized metric vector spaces: $V(\mathbf{C}; t)$. This result holds, irrespective of the space dimension and parameterization structure.

It shall be stressed that in order to accomplish this task there are necessary some properties, which can be described, if preferred, as axioms, concerning the functions of the parameter structure, which constitute in turn the elements of the vectors in these spaces. This connects the present discussion with the well-known specific properties, which quantum wave functions should possess.

Thus, if Heisenberg's relations have to be deduced from a parameterized metric vector space structure, the parametric vector components shall have to be submitted to some axioms. However, one can also conversely admit Heisenberg's relations induce some previous characteristic properties into the metric spaces elements, where quantum systems have to be described.

Therefore, it seems not surprising that Heisenberg's relations constitute a fundamental quantum mechanical cornerstone, connecting the geometry of the parameterized metric spaces with the notions of variance and uncertainty or standard deviation, its square root.

It can be finally said that the constant triad companions covariance, or the same property held by conjugated pairs of quantum mechanical variables, appears to be, in this way, within parameterized vector spaces and, by extension, in quantum systems description too, a formulation closely related to Heisenberg's relations.

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